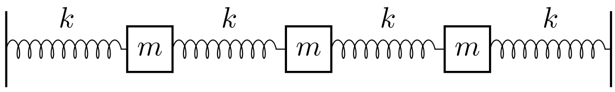
Corrections to `ghm22@cam.ac.uk`*. Star ($\star$) indicates a harder question.*

1 For a system of small oscillations with kinetic energy $T = x_i T_{ij} x_j$ and potential energy $V = x_i V_{ij} x_j$, describe how to find the normal modes $\{\mathbf{Q}_i\}$ and their normal frequencies $\{\omega_i\}$. What is the orthogonality relation between different normal modes? What does a normal frequency of $\omega = 0$ correspond to?

2

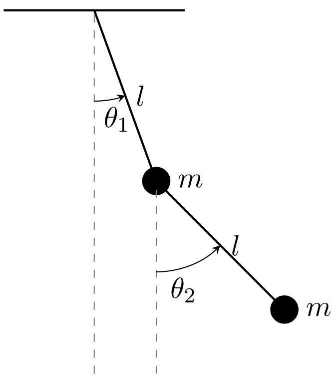


Three equal masses are connected by equal springs, the walls being fixed. Assume the motion is constrained to be 1-dimensional along the line of the springs. Find the normal modes of oscillation and the ratios of the normal frequencies. Are there any periodic solutions other than the pure normal modes?

3 Consider a model of a linear molecule AAB where the atom B is at one end. The masses are $M_A = m$ and $M_B = 2m$. Let the spring constants for the forces between neighbouring atoms be given by $k_{AA} = k$ and $k_{AB} = 2k$. Find the equations of motion and the normal frequencies for linear oscillations along the axis of the molecule. Verify the orthogonality relation for the normal mode vectors $Q^{(m)}$, and find the normalized form of these generalized eigenvectors.

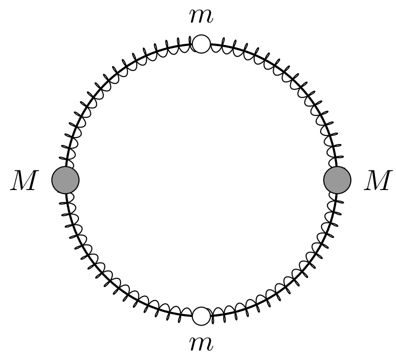
At $t = 0$ all atoms are initially at their equilibrium positions, the atoms A are at rest, and atom B is given an initial velocity u_0. Derive expressions for the positions of the 3 molecules as functions of t, and in terms of the normal modes.

4



Consider the double pendulum shown above, with two balls of equal mass m at the ends of identical rods, length l, whose masses can be neglected. The motion is constrained to be in a fixed vertical plane. Find the two normal modes of small oscillations and their frequencies.

5

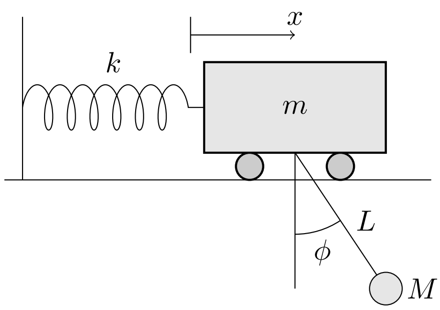


Four beads are threaded on a smooth circular wire of radius R. Their masses alternate as M, m, M, m. They are connected by four identical springs of spring constant k, which lie along the arc of the wire.

Let the angular displacements from their equilibrium positions be $\theta_1, \theta_2, \theta_3, \theta_4$. Show that one normal frequency is zero and explain its physical significance. Find the remaining three normal frequencies and describe the motion of the masses in the highest frequency mode.

6 Let $T = \frac{1}{2} T_{ij} \dot{q}_i \dot{q}_j$ and $V = \frac{1}{2} V_{ij} q_i q_j$. Verify that the equations of motion $T_{ij} \ddot{q}_j + V_{ij} q_j = 0$ imply that the energy $T + V$ is conserved. Does the constancy of $T + V$ suffice to deduce the equations of motion?

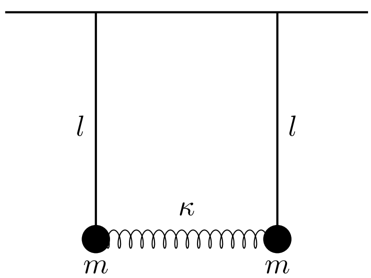
7



A pendulum of mass M and length L is suspended from a cart of mass m that can oscillate on the end of a spring of force constant k, as shown in the figure. The cart is constrained to move in the horizontal direction only, and has a displacement $x(t)$ from its equilibrium position. The pendulum oscillates in the plane making angle $\phi(t)$ with the vertical direction.

Assuming that the angle ϕ and displacement x remain small, write down the system's Lagrangian, and thus find the normal modes.

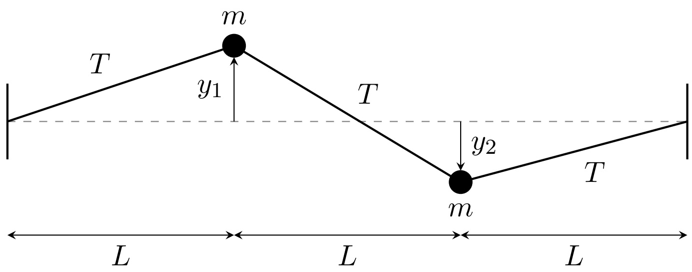
8



Two identical simple pendulums, each of mass m and length l, are suspended from a rigid ceiling. Their bobs are connected by a light spring of spring constant κ, whose natural length equals the distance between the suspension points.

Find the normal modes and frequencies of the system for small oscillations in the plane of the pendulums. If the system is started from rest with the left pendulum displaced by a small angle θ_0 and the right pendulum at its equilibrium position, describe the subsequent motion. What happens in the limit where the spring is very weak ($\kappa \ll mg/l$)?

9



A light, elastic string is stretched with a large, constant tension T between two fixed walls separated by a distance $3L$. Two particles, each of mass m, are attached to the string at distances L and $2L$ from the left wall.

Assuming gravity is negligible and the tension T remains constant for small displacements, write down the potential energy for small transverse displacements y_1 and y_2. Find the normal frequencies and sketch the normal modes of the system.

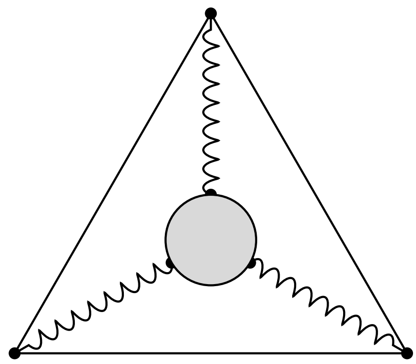
10 A particle of mass m moves in a two-dimensional potential given by

$$V(x, y) = V_0 \left(\frac{x^2}{a^2} + \frac{y^2}{a^2} - 2\frac{x^2 y}{a^3} + \frac{x^4}{a^4} \right)$$

where V_0 and a are positive constants.

Find the position of the stable equilibrium point. By Taylor expanding the potential around this equilibrium point, determine the T_{ij} and V_{ij} matrices for small oscillations. Hence, find the normal modes and their corresponding angular frequencies.

11⋆ A uniform disk of mass m and radius a lies on a horizontal, frictionless surface. It is attached symmetrically to three ideal, massless springs with spring constant k, which are in turn connected to three corners of a fixed, immovable equilateral triangle. In equilibrium, the length of all springs l is greater than their unextended length at rest l_0.

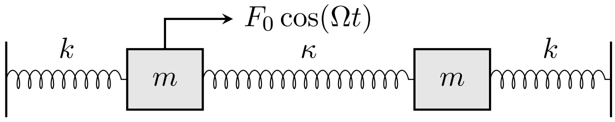


Find the normal modes and normal frequencies of oscillation of the disk.

How do the answers change if the triangle is replaced by an N-gon with N springs attached to the corners?

[Hint: Consider the problem carefully before calculating. Are any of the normal modes degenerate?]

12⋆



Two masses m are connected to fixed walls by springs of constant k, and to each other by a spring of constant κ. The left mass is subjected to an external driving force $F(t) = F_0 \cos(\Omega t)$.

Find the steady-state amplitudes of both masses as a function of the driving frequency Ω. At what frequencies does resonance occur? For what value of Ω is the left mass completely stationary in the steady state (an anti-resonance)?

[Hint:]