NST IB: Mathematical Methods III — Sheet 2: Group theory #1

Corrections to `ghm22@cam.ac.uk`. *Star ($\star$) indicates a harder question.*

1 For each of the four algebras below, either show that it is not a group (with respect to the operation $*$), or else verify that it satisfies the group axioms:

a. $\mathbb{R}$ under the operation $a * b = a - b$.

b. $\mathbb{C} \setminus \{0\}$ under the operation $a * b = ab/(a+b)$.

c. $\mathbb{R} \cup \{-\infty\}$ under the operation $a * b = \log\big(e^a + e^b\big)$, where we stipulate as definitions that $e^{-\infty} = 0$ and hence $\log(0) = -\infty$.

d. The set of odd integers, under the operation $a * b = a + b + 1$.

2 Show that the order of a permutation P is the lowest common multiple of the orders of its component cycles. Resolve

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 9 & 7 & 2 & 5 & 8 & 1 & 3 \end{pmatrix}$$

into cycles and find its order. *[Recall that the order of a group element g is the smallest positive integer n such that $g^n = I$.]*

3 A permutation $P \in \Sigma_N$ may be regarded as permuting the components of an N-vector and is represented by a permutation matrix. For example, the permutation (12)(34) is represented by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Show that the determinant of a permutation matrix is equal to ± 1, and show that the permutations whose matrices have determinant $+1$ form a subgroup of Σ_N with $\frac{1}{2}N!$ elements. Verify that this is the subgroup of even permutations.

4 Use the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{pmatrix}$$

to show that the permutations with cycle decompositions (123)(45) and $(abc)(de)$ are in the same conjugacy class within S_5. Generalize this example to show that two non-identity permutations are in the same conjugacy class within Σ_5 if and only if their cycle decompositions have the same cycle shape. Deduce that there are 7 conjugacy classes in Σ_5.

Based on this, what is the maximum order of an element of Σ_5? What about Σ_9?

5 Show that the symmetries of a regular tetrahedron in 3-dimensional space, including reflections, form a group isomorphic to the permutation group Σ_4 (also denoted S_4). Show that the symmetry group without reflections, i.e. the rigid rotations of a tetrahedron, is isomorphic to the alternating group A_4, the subgroup of Σ_4 consisting of even permutations only.

6 Find the group table for the cyclic group C_4, which is the group of rotations in a plane by angles $\frac{n\pi}{2}$ where $n = 0, 1, 2$ or 3. Find the group table for the so-called Vierergruppe V (also referred to as Klein's four-group and denoted K_4), which consists of the identity and the rotations by π about the x, y, and z axes in 3-dimensional space. Show that both C_4 and V are Abelian, but that they are not isomorphic to each other.

7 Given a group G, what is the conjugacy class of the element $a \in G$?

Prove that these three definitions of a normal subgroup H in G are equivalent:

a. for all $g \in G$ and $h \in H$, $ghg^{-1} \in H$,

b. H is a union of complete conjugacy classes of G,

c. $gH = Hg$ where gH denotes a left coset of H.

8 The six functions $f_1, f_2, \ldots, f_6$ are defined by:

$$f_1(z) = z, \qquad f_4(z) = \frac{1}{z}$$

$$f_2(z) = \frac{1}{1-z}, \qquad f_5(z) = 1 - z$$

$$f_3(z) = 1 - \frac{1}{z}, \qquad f_6(z) = \frac{z}{z-1}$$

Show that these form a group under function composition, i.e. function of a function: the group 'product' operation $f_1 f_2$ is defined by $(f_1 f_2)(z) \equiv f_1(f_2(z))$.

Construct the group table, find all subgroups and determine which of them are normal. You should find that there are three order-2 subgroups, none of which are normal, and one order-3 normal subgroup. Is this group isomorphic to any other group mentioned in the lectures? How do each of the functions $f_1, f_2, \ldots, f_6$ act on the points $z = 0, 1, \infty$?

9 The symmetry group of an N-gon (the *dihedral group* D_n) is generated by a single rotation R by an angle $2\pi/N$ and a reflection m (being any single one of the possible reflection symmetries). Show by means of sketches that the relations $R^N = I$, $m^2 = I$ and $Rm = mR^{-1}$ are always obeyed. This is called the *presentation* of the group:

$$D_n = \langle R, m \,|\, R^N = m^2 = I, RmR = m \rangle \,.$$

Deduce that the elements of the group are R^n and $R^n m$, where $n \in \{1, 2, \ldots, N\}$. Use these relations to express the product of two arbitrary elements of the group, either in the form R^n or $R^n m$.

Describe the conjugacy classes of D_n, carefully distinguishing between the case where n is even and odd. Hence or otherwise, show that D_n has an order 2 normal subgroup if n is even, and not if n is odd.