NST IB: Mathematical Methods II — Sheet 1: Sturm–Liouville Theory

Corrections to `ghm22@cam.ac.uk`*. Star ($\star$) indicates a harder question.*

1 Put the following operators in Sturm–Liouville form by finding an appropriate weight function:

a. $\tilde{\mathcal{L}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x}$

b. $\tilde{\mathcal{L}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}$

2 The function $y(x)$ satisfies the equation $y'' + 2\gamma y' + (\gamma^2 + n^2)y = 0$, for real constant γ and integer n. Show that the boundary conditions $y(0) = y(\pi) = 0$ allow a solution y_n for each n, that is unique up to multiplication by a constant. Why is it sufficient to consider only positive integers? Put the equation for y into Sturm-Liouville form and hence find the function $w(x)$ such that

$$\int_0^{\pi} y_m(x)y_n(x)w(x)\,\mathrm{d}x = 0 \quad \text{for } m \neq n.$$

3 Show that the operator

$$\mathcal{L} = -(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x}$$

is self-adjoint, for inner product with unit weight function, when acting on functions $y(x)$ that are finite at $x = -1$ and $x = 1$. The eigenfunctions of $\mathcal{L}$ with normalization $y(1) = 1$ are the Legendre Polynomials $P_l(x)$ with eigenvalues $l(l+1)$ for non-negative integer l. Verify that

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2 - 1), \quad P_3 = \frac{1}{2}(5x^3 - 3x),$$

and that these eigenfunctions are orthogonal.

Using the orthogonality of eigenfunctions with distinct eigenvalues, show that

$$\int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)\,\mathrm{d}x = 0$$

for $m \neq n$.

Find the solution of the equation $\mathcal{L}y = x^3$ with the above boundary conditions, in terms of P_1 and P_3. What happens if x^3 is replaced by $x^2 - k$ with (i) $k = \frac{1}{3}$ and (ii) $k \neq \frac{1}{3}$?

4 Consider the periodic Sturm–Liouville problem defined by the operator $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the interval $[-\pi, \pi]$ subject to the periodic boundary conditions $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$. Find the eigenvalues and a complete set of real-valued eigenfunctions. Explain how expanding an arbitrary periodic function $f(x)$ in terms of these eigenfunctions yields the standard Fourier series.

5 The Hermite differential equation is given by $y'' - 2xy' + 2\lambda y = 0$. Transform this equation into Sturm–Liouville form and identify the weight function $w(x)$ on the interval $(-\infty, \infty)$.

Assuming a power series solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$, find the recurrence relation for the coefficients a_k. Show that for $\lambda = n$ (a non-negative integer), the series terminates to give a polynomial solution $H_n(x)$ (the Hermite polynomials). Find explicit polynomial expressions for H_0, H_1, and H_2 assuming the normalization where the coefficient of x^n is 2^n.

6 Find the eigenvalues of the differential operator $\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$ acting on functions $y(x)$ subject to the boundary conditions $y(0) = y'(\pi) = 0$. Write down the Green's function for $\mathcal{L}$ in terms of its orthonormal eigenfunctions. Find the eigenfunction expansion of the function $f(x) = x(2\pi - x)$. Hence find a solution, as a sum over eigenfunctions, of the equation $y'' - y = x(x - 2\pi)$ subject to the above boundary conditions.

7 Transform the differential equation $x^2y'' + 3xy' + \lambda y = 0$ into Sturm–Liouville form on the interval $[1, e]$. Find the appropriate weight function $w(x)$. Given the boundary conditions $y(1) = y(e) = 0$, explicitly solve the equation to find the eigenvalues λ_n and corresponding eigenfunctions $y_n(x)$, and verify their orthogonality with respect to $w(x)$.

8 For a regular Sturm–Liouville problem $\mathcal{L}y = \lambda w(x)y$ on $[a, b]$ with $\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + q(x)$ and $y(a) = y(b) = 0$, the Rayleigh quotient is defined as

$$R[y] = \frac{\langle y, \mathcal{L}y\rangle}{\langle y, y\rangle_w} = \frac{\int_a^b \left[p(y')^2 + qy^2\right] \mathrm{d}x}{\int_a^b wy^2 \, \mathrm{d}x}.$$

Derive the integral form from the inner product form.

Consider the equation $-y'' = \lambda y$ on $[0, 1]$ with $y(0) = y(1) = 0$. Use the trial function $y_{\text{trial}}(x) = x(1 - x)$ to estimate the lowest eigenvalue λ_1 using the Rayleigh quotient, and compare to the exact value.

9 Let $\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)$ be a regular Sturm–Liouville operator on $[a, b]$ with $p(x) > 0$ and weight function $w(x) > 0$.

a. Prove that if $q(x) \geq 0$ for all $x \in [a, b]$, then any eigenvalue λ associated with the Dirichlet boundary conditions $y(a) = y(b) = 0$ must be strictly positive ($\lambda > 0$).

b. What can you say about the eigenvalues if $q(x) = 0$ everywhere and the boundary conditions are changed to Neumann conditions $y'(a) = y'(b) = 0$? Find the corresponding ground-state eigenfunction.

10⋆ The general real 4th-order linear differential operator acting on functions of the real variable may be written as

$$\mathcal{L} = p(x)\frac{\mathrm{d}^4}{\mathrm{d}x^4} + q(x)\frac{\mathrm{d}^3}{\mathrm{d}x^3} + r(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + s(x)\frac{\mathrm{d}}{\mathrm{d}x} + t(x)$$

for real functions p, q, r, s, t. Find necessary conditions on these functions such that $\mathcal{L}$ is self-adjoint for inner product with (i) unit weight function, and (ii) arbitrary weight function $w(x)$. Is there always a choice of w that makes $\mathcal{L}$ self-adjoint?