Corrections to `ghm22@cam.ac.uk`. *Star ($\star$) indicates a harder question.*

The first two questions are revision of Lagrange multipliers from 1A course B. Feel free to skip them if you're confident. Likewise, if confident, there are some harder questions at the end!

1 The temperature of a point on the unit sphere $x^2 + y^2 + z^2 = 1$ is given by $T(x, y, z) = 1 + xyz$. Find the points of maximum and minimum temperature on the sphere. What are their temperatures?

2 A coal box, in the shape of a cuboid, is to be placed flush against a wall so that only its top, front and two ends are visible. How should the height h and the depth d be chosen so as to minimize the visible surface area A under the constraint that the box has a given volume V?

3 An area A of a field is enclosed by a length l of flexible fencing with its ends attached a distance a apart on a straight wall, where $a < l < \frac{1}{2}\pi a$. What shape maximizes A?

4 Extremize the functional

$$F[y] = \int_0^a (y'^2 + yy' + 2y)\,\mathrm{d}x$$

with the boundary conditions $y(0) = y(a) = 0$ and subject to the constraint $\int_0^a y\,\mathrm{d}x = l$ where l is a constant.

5 When the integrand $f(y, y')$ does not explicitly depend on the independent variable x, show that the Euler–Lagrange equation admits the first integral

$$f - y'\frac{\partial f}{\partial y'} = C$$

where C is a constant.

A heavy uniform rope of length L hangs under gravity between two fixed points $(-a, 0)$ and $(a, 0)$. By minimizing the total potential energy functional

$$E[y] = \rho g \int_{-a}^{a} \mathrm{d}l = \rho g \int_{-a}^{a} y\sqrt{1 + y'^2}\,\mathrm{d}x$$

subject to the fixed length constraint $\int_{-a}^{a} \sqrt{1 + y'^2}\,\mathrm{d}x = L$, use the first integral to show that the shape of the rope is a catenary $y(x) = c + \lambda\cosh\left(\frac{x}{\lambda}\right)$, where c and λ are constants.

6 A soap film is bounded by two circular wires $r = a$, $z = \pm b$ in cylindrical polar coordinates (r, θ, z). Given that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is,

$$r = c\cosh(z/c),$$

where c satisfies the condition $a/c = \cosh(b/c)$. Show graphically that this condition has no solution for c if b/a is larger than a certain critical ratio. What happens to the soap surface as b/a is increased from below this ratio to above it?

7 Show from first principles that the functional $I[x] = \int_{t_1}^{t_2} f(t, x, \dot{x}, \ddot{x})\,dt$ is extremized, for variations with both $x(t)$ and $\dot{x}(t)$ fixed at t_1 and t_2, by solutions of the equation

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0$$

Hence find the function $x(t)$ with $x(1) = 1$, $\dot{x}(1) = -2$, $x(2) = 1/4$ and $\dot{x}(2) = -1/4$ that minimizes $\int_1^2 t^4\{\ddot{x}(t)\}^2\,\mathrm{d}t$.

8 State Fermat's principle governing the paths traced by light rays in a medium. A horizontally stratified medium has refractive index $\mu(z) = \sqrt{a - bz}$, where z is height and a and b are positive constants. Prove that the path of a light ray within a vertical plane in this medium is an inverted parabola.

9 Given that the refractive index $\mu(\mathbf{r})$ of some material equals $|\nabla f|$ for some function $f(\mathbf{r})$, show that the optical path length $\int_A^B \mu \, dl$ between points A and B in the material is no less than $f(B) - f(A)$, with equality if and only if the path is orthogonal to the family of surfaces of constant f. Deduce that such 'orthogonal' trajectories satisfy Fermat's principle.

10 The differential equation governing small transverse displacements $y(x)$ of a string with fixed endpoints at $x = 0$ and $x = \pi$ is,

$$y'' + \omega^2 f(x) y = 0,$$

where ω is the angular frequency of the vibration and f is a positive function. Show that the allowed values of ω^2 are given by the stationary values of the ratio of functionals $\Omega^2 = F/G$, where $F = \int_0^\pi (y')^2 \, \mathrm{d}x$ and $G = \int_0^\pi f y^2 \, \mathrm{d}x$. Use this fact to find an approximate value for the angular frequency of the fundamental mode when $f = 1 + \epsilon \sin x$ assuming $\epsilon \ll 1$.

11 A Sturm–Liouville eigenvalue problem for functions $y(x)$ satisfying $y(\pm 1) = 0$ has the equation

$$-(1 + x^2) y'' - 2xy' = \lambda y.$$

Use the Rayleigh–Ritz method with trial function $y_1 = 1 - x^2$ to obtain an upper bound on the lowest eigenvalue λ. Show that a better bound is obtained from the trial function $y_2 = \cos(\pi x/2)$. Explain how a further improvement could be achieved by considering y_1 and y_2 in combination.

[Hint: $\int_{-1}^{1} x^2 \sin^2(\pi x/2) \, \mathrm{d}x = 1/3 + 2/\pi^2$]

12 Show that $\psi_0 = \exp\left(-\frac{1}{2}x^2\right)$ is an eigenfunction of the operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x^2 - 1)$$

acting on functions $\psi(x)$ for which $\psi \to 0$ as $|x| \to \infty$ and find the corresponding eigenvalue λ_0. This is in fact the lowest eigenvalue. Use the Rayleigh–Ritz method with trial function

$$\tilde{\psi}_0 = \begin{cases} b(a^2 - x^2) & |x| < a \\ 0 & |x| \geq a \end{cases}$$

where a and b are adjustable parameters, to obtain an approximation $\tilde{\lambda}_0$ to λ_0. Comment on the sign of $\tilde{\lambda}_0 - \lambda_0$.

13⋆ For the Sturm–Liouville system $-y'' = \lambda y$ with $y(0) = y(\pi) = 0$, the ground state eigenvalue is $\lambda_0 = 1$ with eigenfunction $\psi_0(x) = \sin x$. Suppose we wish to estimate the first excited eigenvalue λ_1 using the Rayleigh–Ritz method with a trial function of the form $y(x) = x(\pi - x)(1 + Ax)$. Determine the value of the constant A such that $y(x)$ is orthogonal to $\psi_0(x)$ with respect to the weight function $w(x) = 1$. Use this trial function to compute an upper bound for λ_1, and compare it to its exact value.

14⋆ Let Ω be a bounded region in $\mathbb{R}^2$ with boundary $\partial\Omega$. Consider the functional

$$E[u] = \iint_\Omega \left(\frac{1}{2} |\nabla u|^2 - f(x, y) u \right) \mathrm{d}x \, \mathrm{d}y$$

where $u(x, y)$ is fixed to a given function $g(x, y)$ on the boundary $\partial\Omega$. By calculating the variation δE, show that the function u which minimizes $E[u]$ satisfies Poisson's equation $\nabla^2 u + f = 0$ in Ω. This is an example of *Dirichlet's principle*.