NST IB: Mathematical Methods II — Sheet 3: General solutions to PDEs

Corrections to `ghm22@cam.ac.uk`. *Star ($\star$) indicates a harder question.*

1 A solute with diffusivity D, of concentration $\Phi(r,\phi)$ in plane polar coordinates, both surrounds and fills a cylinder of radius a. The boundary of the cylinder is permeable and the solute diffuses through it in such a way that the flux has normal component $F\cos 2\phi$ at $r = a$ where $Ffestive$ is a constant. Assuming a steady state, find the distribution of solute in the interior of the cylinder. Why does your answer include an arbitrary constant? How is it possible for there to be a constant flux of solute through the boundary of the cylinder and yet for the concentration within to remain fixed?

2 A single-valued potential $\Phi(r,\phi)$, in plane polar coordinates, satisfies $\nabla^2\Phi = 0$ inside a cylindrical annulus $a < r < b$. On $r = b$, $\Phi = 0$ for all ϕ. On $r = a$, $\Phi = 0$ for $-\pi < \phi \leq 0$ and $\Phi = 1$ for $0 < \phi \leq \pi$. Find $\Phi(r,\phi)$ inside the annulus.

3 The axis of a solid cylinder of radius a coincides with the z-axis of cylindrical polar coordinates (r,ϕ,z). Heat flows past the cylinder such that the temperature T inside and outside, i.e. $r < a$ and $r > a$, is independent of z and satisfies $\nabla^2 T = 0$. Find T subject to the following boundary conditions: T finite at $r = 0$ and $T \sim Gr\cos\phi$ as $r \to \infty$, where G is a constant; T continuous at $r = a$ but $\frac{\partial T}{\partial r} = \beta\frac{\partial T}{\partial r}$ for constant β $(0 < \beta < 1)$. What is the physical meaning of the constants G and β? [a^+ denotes the limit $r \to a$ from $r > a$ and a^- denotes the limit $r \to a$ from $r < a$.]

4 A potential $\Phi(r,\phi)$ satisfies the two-dimensional Poisson equation $\nabla^2\Phi = -\rho_0/\epsilon_0$ inside a circular disc $0 \leq r < a$, where ρ_0 is a uniform constant charge density. The boundary of the disc is held at a potential $\Phi(a,\phi) = V_0\cos^3\phi$. By seeking a particular solution that depends only on r, find the complete solution for $\Phi(r,\phi)$ inside the disc.

5 Find the solution to $\nabla^2\Psi = 0$ inside a sphere of radius a for axisymmetric Ψ, i.e. $\partial\Psi/\partial\varphi = 0$ in spherical polar coordinates (r,θ,φ), subject to the boundary condition that $\Psi = 1 + \cos\theta + \cos^2\theta$ on the surface of the sphere. Include a derivation of any general solution that you use.

6 In the presence of an electric charge density $\rho_q(r)$, the electrostatic potential $\Phi(r)$ satisfies $\nabla^2\Phi = -\rho_q/\epsilon_0$. In spherical polar coordinates (r,θ,φ), the charge density is

$$\rho_q = \begin{cases} r^{-1}\cos\theta & 0 < r < a \\ 0 & r \geq a \end{cases}$$

Why must both Φ and $\partial\Phi/\partial r$ be everywhere continuous despite the discontinuity in ρ_q at $r = a$? By writing Φ as a function of r times a suitably chosen function of θ and assuming it to be independent of φ, find Φ subject to the boundary conditions that it be finite at $r = 0$ and zero at $r = \infty$. [You may wish to start by proving that the general solution to the differential equation $r^2R'' + 2rR' - 2R = \alpha r$ where α is a constant, is $R = Ar + Br^{-2} + (\alpha/3)r\log r$.]

7 Show, for each of the following three cases, that the solution of the given equation for scalar field Φ in volume V, with surface S, is unique subject to the boundary conditions of the specified type.

(i) Laplace equation: $\nabla^2\Phi = 0$ with normal derivative $\partial\Phi/\partial n$ specified on S (Neumann boundary conditions) and Φ specified at one point.

(ii) Yukawa equation: $\nabla^2\Phi = m^2\Phi$ for real non-zero constant m, with Φ specified on S (Dirichlet boundary conditions).

(iii) Yukawa equation subject to Neumann boundary conditions.

8 The potential $\Phi(x, y)$ satisfies Laplace's equation $\nabla^2\Phi = 0$ inside the rectangular domain $0 \le x \le a$, $0 \le y \le b$. The boundary conditions are given by $\Phi(0, y) = 0$, $\Phi(a, y) = 0$, $\Phi(x, 0) = 0$, and $\Phi(x, b) = f(x)$. Use separation of variables to find the general solution for $\Phi(x, y)$. Evaluate the coefficients explicitly for the case where $f(x) = T_0$, where T_0 is a constant. Why does the solution fail to converge uniformly at the corners $(0, b)$ and (a, b)?

9 A two-dimensional wedge-shaped region is defined in plane polar coordinates (r, ϕ) by $0 \le r \le a$ and $0 \le \phi \le \alpha$, where $\alpha < 2\pi$. The potential $\Phi(r, \phi)$ satisfies $\nabla^2\Phi = 0$ within the wedge, subject to the boundary conditions $\Phi(r, 0) = 0$, $\Phi(r, \alpha) = 0$, and $\Phi(a, \phi) = V_0$, where V_0 is a constant. Find $\Phi(r, \phi)$ inside the wedge. Examine the behavior of the electric field $\mathbf{E} = -\nabla\Phi$ near the corner $r \to 0$, and comment on the physical significance of the cases $\alpha < \pi$ and $\alpha > \pi$.

10 Consider the solution of $\nabla^2\Phi = 0$ inside a solid hemisphere of radius a defined by $0 \le r \le a$ and $0 \le \theta \le \pi/2$ in spherical coordinates. The flat base ($\theta = \pi/2$) is held at zero potential, while the curved surface ($r = a$) is held at a constant potential Φ_0.

Write down the general solution in Legendre polynomials which exists at $r = 0$. Explain why there will be no even order Legendre polynomials present in the solution.

Harder: Find the general solution in terms of Legendre polynomials over the half-period $[0, \pi/2]$.

You may assume without proof that the Legendre polynomials satisfy $P_n(0) = \frac{(-1)^{n/2} n!}{2^n((n/2)!)^2}$ for even n, the recurrence relation $(2n+1)P_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[P_{n+1}(x) - P_{n-1}(x)\right]$, and the orthogonality result,

$$\int_0^1 P_n(x)P_m(x)\,\mathrm{d}x = \frac{1}{2n+1}\delta_{nm} \quad \text{for odd } n, m.$$