Corrections to `ghm22@cam.ac.uk`*. Star* ($\star$) *indicates a harder question.*

1 The quarter-plane $x > 0$, $y > 0$ is occupied by a single source of heat of strength Q positioned at the point (x_0, y_0). At $x = 0$ there is a plane conducting wall held at temperature T_0. At $y = 0$ there is an insulated wall across which no heat can flow (i.e. the heat flux normal to the wall must vanish). Write down the equation and boundary conditions satisfied by a time-independent temperature field $T(x, y)$ and use the method of images to find it. Hence show that the magnitude of the heat flux across the wall at the point $(0, y)$ is

$$\frac{Qx_0}{\pi}\left\{\frac{1}{x_0^2+(y-y_0)^2}+\frac{1}{x_0^2+(y+y_0)^2}\right\}.$$

Calculate the total heat radiated across the wall at $x = 0$ and comment on the result.

2 Define the Green's function $G(\mathbf{r}, \mathbf{r}')$ for the three-dimensional Laplacian with Dirichlet boundary conditions, acting on functions defined in a volume V with bounding surface S. Given that $\nabla^2 u = 0$ in V and that $u = f$ on S, show that

$$u(\mathbf{r}') = \int_S f(\mathbf{r})\frac{\partial G}{\partial n}\,\mathrm{d}S$$

where $\partial/\partial n$ denotes differentiation along the outward normal on S. What is the analogous equation for two space dimensions? Show that a solution of $\nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}')$ in two dimensions is $G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi}\log|\mathbf{r} - \mathbf{r}'|$. Hence find the Green's function, with Dirichlet boundary conditions, for the two-dimensional Laplacian in the half-plane $-\infty < x < \infty$, $y > 0$. Use it to solve the Laplace equation for u in this region with boundary conditions that $u \to 0$ as $r \to \infty$, $u = 0$ on $y = 0$ for $|x| > 1$, and $u = 1$ on $y = 0$ for $|x| \le 1$.

3 A thin disc of uniform density and total mass M lies in the plane $z = 0$ of cylindrical polar coordinates (r, ϕ, z) and occupies the region $r \le a$ of this plane. Use the integral expression for a solution of Poisson's equation to find the gravitational potential $\Phi(r, z)$ on the axis of symmetry $r = 0$. Assuming that $|z| \gg a$, expand your result as a power series accurate to $\mathcal{O}(z^{-5})$. Now write down the general solution of Laplace's equation in spherical polar coordinates valid at large distance r_s from the origin. By comparing this with your previous expansion, find an expression for $\Phi(r, \theta)$ that is valid off the axis of symmetry. *[Hint: Recall that* $P_n(\cos\theta) = 1$ *when* $\cos\theta = 1$*.]*

4 State a version of Green's identity applicable to a plane surface S bounded by a closed curve C. Use this identity, and the Green's function from Question **??**, to show that the solution of $\nabla^2\Phi = 0$ within the disc of radius a, i.e. $r < a$ in plane polar coordinates (r, ϕ), subject to the boundary condition that $\Phi = \Psi(\phi)$ on the boundary at $r = a$, is

$$\Phi(r,\phi) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{\Psi(\phi')\,\mathrm{d}\phi'}{a^2-2ar\cos(\phi-\phi')+r^2}.$$

5 Use the result of Question 4 to show that in $0 \le r < 1$, and for $F(r, \phi, \phi') \equiv 1-2r\cos(\phi - \phi')+r^2$:

$$\int_0^{2\pi}\frac{\mathrm{d}\phi'}{F(r,\phi,\phi')} = \frac{2\pi}{1-r^2}$$

$$\int_0^{2\pi}\frac{\sin(\phi')\,\mathrm{d}\phi'}{F(r,\phi,\phi')} = \frac{2\pi r\sin\phi}{1-r^2}$$

$$\int_0^{2\pi}\frac{\cos^2(\phi')\,\mathrm{d}\phi'}{F(r,\phi,\phi')} = \frac{2\pi r^2\cos^2\phi}{1-r^2}+\pi.$$

6 Find the spherically symmetric, free-space Green's function $G(\mathbf{r}, \mathbf{r}')$ for the three-dimensional Helmholtz equation

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}').$$

By substituting $G = f(R)/R$, where $R = |\mathbf{r} - \mathbf{r}'|$, and keeping only the solution proportional to e^{ikR}, show that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

[Hint: Consider integrating the equation over a small sphere centered at $R = 0$ to fix the normalization constant.]

7 Consider the Helmholtz equation $(\nabla^2 + k^2)u = 0$ in the half-space $z > 0$, subject to the inhomogeneous Dirichlet boundary condition $u(x, y, 0) = f(x, y)$ on the plane $z = 0$. Use the method of images and the free-space Green's function from the previous question to construct the appropriate Dirichlet Green's function $G_D(\mathbf{r}, \mathbf{r}')$ for this domain. Hence, show that the solution can be written in the form

$$u(\mathbf{r}') = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(x, y) \left(\frac{ik}{R} - \frac{1}{R^2} \right) \frac{z'}{R} e^{ikR} \, \mathrm{d}x \, \mathrm{d}y \,,$$

where $R^2 = (x - x')^2 + (y - y')^2 + z'^2$.

8 Find the Green's function for the Laplacian in the infinite two-dimensional strip $-\infty < x < \infty$, $0 < y < L$, subject to Dirichlet boundary conditions $G = 0$ on the parallel walls $y = 0$ and $y = L$. *[Hint: You may find it helpful to use a Fourier transform in the x-direction while maintaining an explicit differential structure in the y-direction.]*