

Corrections to *ghm22@cam.ac.uk*. Star ( $\star$ ) indicates a harder question.

- 1 Let  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  be a set of orthonormal basis vectors. Confirm that the vectors

$$\mathbf{e}'_1 = \frac{1}{2}(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_3), \quad \mathbf{e}'_2 = \mathbf{e}_2, \quad \mathbf{e}'_3 = \frac{1}{2}(\mathbf{e}_3 - \sqrt{3}\mathbf{e}_1)$$

are orthonormal and hence define a new frame. If  $\mathbf{v} = \sqrt{3}\mathbf{e}_1 + 5\mathbf{e}_2 + 2\mathbf{e}_3$ , find its components,  $v'_i$  in the new frame. The moments of inertia of a rigid body, with respect to its principal axes  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ , are 1, 2 and 3. Find the inertia tensor  $I'$  in the frame defined by  $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ . With respect to a third (orthonormal) frame  $(\mathbf{e}''_1, \mathbf{e}''_2, \mathbf{e}''_3)$ , the vector  $\mathbf{e}_1$  has components  $\frac{1}{\sqrt{2}}(1, 1, 0)$  and the vector  $\mathbf{e}_2$  has components  $\frac{1}{\sqrt{3}}(1, -1, 1)$ . Find the inertia tensor  $I''$  in this frame. Compute the trace of the square of  $I''$  and verify that it agrees with the trace of the square of  $I$ ; why must this be so? [The moments of inertia are the diagonal entries of the inertia tensor  $I$  in the frame provided by the (orthonormal) principal axes; the off-diagonal entries are zero in this frame.]

- 2 A right circular solid cylinder of uniform mass density  $\rho$  has radius  $a$  and length  $\sqrt{3}a$ . Identify the principal axes of this rigid body and in the frame defined by these axes find its inertia tensor about its centre of mass. Comment on your result.
- 3 The components of a vector  $\mathbf{F}$  are related to those of vectors  $\mathbf{J}$  and  $\mathbf{H}$  by an equation of the form  $F_i = T_{ijk}J_jH_k$  for arbitrary  $\mathbf{J}$  and  $\mathbf{H}$ . Starting from the transformation law for the components of a vector, deduce that the coefficients  $T_{ijk}$  are the components of a 3rd-order tensor.
- 4 An isotropic tensor is defined as a tensor whose components are identical in all rotated coordinate systems. Prove that:
- There are no non-zero isotropic first-order tensors (vectors).
  - The only isotropic second-order tensors are scalar multiples of the Kronecker delta  $\delta_{ij}$ . [Hint: Consider the effect of a  $90^\circ$  rotation about one of the coordinate axes, and then a  $45^\circ$  rotation.]
- 5 Show that any second-order tensor  $T$  may be expressed, uniquely, as the sum of a symmetric tensor  $S$  with zero trace, an isotropic tensor  $I$  and an antisymmetric tensor  $A$ . Exhibit this decomposition for the tensor whose components are

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 5 \\ 1 & 0 & 5 \\ 1 & -5 & 3 \end{pmatrix}$$

- 6 Let  $A$  be an antisymmetric 2nd-order tensor, with components  $A_{ij}$  in a frame in which the vector  $\mathbf{x}$  has components  $x_i$ . Show that the vector  $\mathbf{Ax}$ , with components  $A_{ij}x_j$ , may be written as  $\boldsymbol{\omega} \times \mathbf{x}$  for some vector  $\boldsymbol{\omega}$  determined by  $A$ . Now show that the tensor  $B = A^2$  is symmetric, and find its eigenvalues in terms of  $\boldsymbol{\omega}$ .
- 7 The electrical conductivity tensor  $\sigma_{ij}$  determines the electric current density  $\mathbf{J}$  that flows in response to a uniform and constant applied electric field  $\mathbf{E}$ , according to the formula  $J_i = \sigma_{ij}E_j$ . The electrical conductivity tensor of a crystal is measured to have entries

$$\sigma_{ij} = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

in a laboratory frame. Show that there is a direction in which no current flows, and find that direction. The rate of energy dissipation per unit volume is given by  $\mathbf{E} \cdot \mathbf{J}$ . For  $|\mathbf{E}|^2 = 1$  find the minimum and maximum possible energy dissipation rates.

- 8 The eigenvalues of the conductivity tensor  $\sigma_{ij}$  of an anisotropic crystal are  $\alpha$ ,  $\beta$  and  $\gamma$ . Write down the tensor in the frame in which it is diagonal and identify the axis of symmetry due to a repeated eigenvalue (e.g., if  $\alpha = \beta$ ). A particular crystal is grown such that it takes the form of a long thin wire of length  $L$  and cross-sectional area  $A$ , the direction of the wire making an angle  $\theta$  with the axis of symmetry of the crystal. Show that the wire's resistance to a current passing along it is

$$R = (L/A)(\alpha^{-1} \cos^2 \theta + \gamma^{-1} \sin^2 \theta).$$

- 9 Let  $M$  be a second-order tensor with components  $M_{ij}$  in one frame and components  $M'_{ij}$  in another frame. Explain (i) why  $\epsilon_{ijk} \det M = \epsilon_{lmn} M_{il} M_{jm} M_{kn}$  and (ii) why  $\epsilon_{ijk}$  are the components of a 3rd-order isotropic pseudo-tensor. (iii) Deduce that  $\det M' = \det M$ , and hence that  $\det M$  is a scalar.

- 10 A fourth-order tensor has components

$$c_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk},$$

where  $\alpha, \beta$  and  $\gamma$  are constants. Show that this defines an isotropic tensor. It is, in fact, the most general fourth-order isotropic tensor, and you may assume this for the rest of the question. In an isotropic fluid the (scalar) pressure  $p$  and the second-order stress tensor  $\sigma$  are related by  $p = -\frac{1}{3} \sigma_{ii}$ . Given that the deviatoric stress tensor  $s$  is defined by  $s_{ij} = \sigma_{ij} + p \delta_{ij}$ , show that it has zero trace. For a fluid with velocity field  $\mathbf{v}(\mathbf{x})$ , the second-order strain tensor  $e$  is defined by

$$e_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

In a "Newtonian" fluid the deviatoric stress tensor  $s$  is linearly related to the strain tensor  $e$  by a fourth-order tensor. Show that the relationship between  $s$  and  $e$  must take the form

$$\sigma_{ij} = 2\mu \left( e_{ij} - \frac{1}{3} \delta_{ij} e_{kk} \right) - p \delta_{ij}$$

for some constant  $\mu$  (known as the coefficient of viscosity). Given that  $\mu > 0$ , show further (by considering an appropriate frame, or otherwise) that the scalar  $s_{ij} e_{ij}$  is non-negative.

- 11 A second-order tensor  $T$  is defined by  $T_{ij} = \delta_{ij} + \epsilon_{ijk} x_k$  where  $x_k$  are the (Cartesian) components of a position vector  $\mathbf{x}$ . Compute the following surface integrals, where in each case the surface of integration is the unit sphere  $|\mathbf{x}| = 1$ :

a.  $\iint x_i \, dS,$

b.  $\iint T_{ij} \, dS,$

c.  $\iint T_{ij} T_{jk} \, dS,$

d.  $\iint n_i n_j n_k n_l \, dS$  where  $n$  is a unit vector.

- 12★ State the Divergence Theorem in suffix notation for a second-order tensor field  $\sigma_{ij}(\mathbf{x})$  defined over a volume  $V$  bounded by a closed surface  $S$  with outward unit normal  $n_i$ .

A continuous medium occupying volume  $V$  is in static equilibrium under the action of a surface traction vector  $t_i = \sigma_{ij}n_j$  and a body force per unit volume  $f_i$ , meaning that  $\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0$  throughout  $V$ . The total torque about the origin exerted on the body is given by  $T_i = \int_S \epsilon_{ijk}x_j t_k dS + \int_V \epsilon_{ijk}x_j f_k dV$ . Show that this total torque can be expressed as

$$T_i = \int_V \epsilon_{ijk} \sigma_{kj} dV.$$

Deduce that local conservation of angular momentum requires the stress tensor  $\sigma_{ij}$  to be symmetric.

13★ In an electrostatic field with electric vector  $E_i$ , the Maxwell stress tensor is defined by

$$\Sigma_{ij} = \epsilon_0 \left( E_i E_j - \frac{1}{2} E_k E_k \delta_{ij} \right),$$

where  $\epsilon_0$  is a constant scalar.

(a) Show that the divergence of this tensor can be written as

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = \epsilon_0 \left( E_i \frac{\partial E_j}{\partial x_j} + E_j \left( \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \right) \right).$$

(b) In a region containing a local charge density  $\rho$  but no time-varying magnetic fields, the governing Maxwell equations are  $\frac{\partial E_j}{\partial x_j} = \frac{\rho}{\epsilon_0}$  and  $\epsilon_{jkl} \frac{\partial E_l}{\partial x_k} = 0$ . Deduce that  $\frac{\partial \Sigma_{ij}}{\partial x_j} = \rho E_i$ . What is the physical interpretation of this result?