NST IB: Mathematical Methods II Sheet 6: Residues & Cauchy's theorem

Corrections to `ghm22@cam.ac.uk`. *Star ($\star$) indicates a harder question.*

This sheet introduces you to the calculus of residues, Cauchy's theorem, and some relatively easy contour integration.

1 Establish the following general methods for calculating residues. [These are all very useful in practice and you are advised to memorize them.]

a. If $f(z)$ has a simple pole at $z = z_0$ then the residue of f at $z = z_0$ ($\text{Res}(f, z_0)$) is given by $\lim_{z \to z_0}\{(z - z_0)f(z)\}$.

b. If $f(z)$ is analytic then the residue of $f(z)/(z - z_0)$ at $z = z_0$ is $f(z_0)$.

c. If $f(z)$ has a simple zero at $z = z_0$ then the residue of $1/f(z)$ at $z = z_0$ is $1/f'(z_0)$.

d. If $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic then the residue of $g(z)/h(z)$ at $z = z_0$ is $g(z_0)/h'(z_0)$.

e. If $f(z)$ has a pole of order N at $z = z_0$ then the residue of $f(z)$ at $z = z_0$ is

$$\lim_{z \to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left[(z - z_0)^N f(z)\right]\right\}.$$

2 Find the poles of the following functions and the residues at each pole:

a. $\dfrac{z+1}{z^2}$

b. $\dfrac{e^{-z}}{z^3}$

c. $\dfrac{\sin^2 z}{z^5}$

d. $\cot z$

e. $\dfrac{z^2}{(1+z^2)^2}$

3 What is a branch point, and a branch cut? Describe how branch cuts can be used to make otherwise ill-defined contour integrals defined.

Sketch possible branch cuts for the following complex functions, giving the values on either side of each cut:

a. $(z^2+1)^{\frac{1}{2}}$

b. $(z^2+1)^{\frac{1}{3}}$

c. $\log\left[\left(\dfrac{z-i}{z+i}\right)^2\right]$

4 State and prove Cauchy's Theorem.

Let C be a closed contour that encloses, in a positive sense, the point z_0 in the complex z-plane, and let n be an integer. Show that

$$\oint_C (z - z_0)^n \,\mathrm{d}z = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1. \end{cases}$$

Given that $f(z)$ is an analytic function, show that,

$$\oint_C \frac{f'(z)\,\mathrm{d}z}{z-z_0} = \oint_C \frac{f(z)\,\mathrm{d}z}{(z-z_0)^2}.$$

5 Let $f(z)$ be a function that is analytic within and on the circle $|z-z_0|=r$ in the z-plane. For non-negative integer n, show that

$$|f^{(n)}(z_0)| \le \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|.$$

Hence prove Liouville's Theorem: a function $f(z)$ that is analytic and bounded for all z is constant. Deduce that any polynomial $p(z)$ of degree at least one has at least one zero.

[Hint: consider $1/p(z)$*.]*

6 Let $f(z)$ be analytic within and on a circle C parametrized by $z = z_0 + re^{i\theta}$ for $0 \le \theta \le 2\pi$.

Use Cauchy's Integral Formula to show that

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,\mathrm{d}\theta\,.$$

Hence, prove the maximum modulus principle: if $f(z)$ is non-constant and analytic on a bounded domain, $|f(z)|$ cannot attain a local maximum at an interior point.

7 Explain how the calculus of residues may be used to evaluate the integral of a function $f(z)$ around a closed contour C in the complex z-plane. Let

$$f(z) = \frac{z^n}{(z-a)(z-a^{-1})}$$

for real constant $a > 1$ and non-negative integer n. By choosing C to be a circle of unit radius, and using the calculus of residues, evaluate the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(n\theta)\,\mathrm{d}\theta}{1-2a\cos\theta + a^2}$$

8 By integrating around a rectangular contour with vertices at $\pm R$ and $i\pi \pm R$, where R is a large real constant, show that $\int_0^\infty \operatorname{sech} x\,\mathrm{d}x = \pi/2$.

9 Let f be an analytic function inside and on a contour C. Assuming f is nonzero aside from an order n zero at z_0, by considering the Laurent expansion of f about that zero, show that,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = n.$$

Let g be a meromorphic (analytic aside from a set of isolated poles) function inside and on C. Extend the argument above to show that,

$$\frac{1}{2\pi i}\oint_C \frac{g'(z)}{g(z)}\,\mathrm{d}z = Z - P$$

where Z is the number of zeros and P is the number of poles of $g(z)$ inside C (counted with multiplicity).

10⋆ Let f be a function that is analytic everywhere in the complex plane $\mathbb{C}$ except at a finite number of isolated poles $z_1, z_2, \ldots, z_n$. Let C_R be a positively oriented (counterclockwise) circle centered at the origin with a radius R large enough to enclose all the poles.

We wish to define the idea of a *residue at infinity* by requiring that the contour integral C_R is the same whether we regard it as enclosing the poles at $\{z_i\}$ or enclosing the single pole at infinity.

Argue that from the perspective of the point at infinity, the path C_R is clockwise, and thus, from Cauchy's theorem,

$$\oint_{C_R} f(z)\underset{.}{z} = -2\pi i \,\mathrm{Res}(f, \infty).$$

Make the substitution $z = 1/w$ to transform the point at infinity to zero. Does this change the orientation of the contour? Argue that the residue at infinity is,

$$\mathrm{Res}(f, \infty) = \mathrm{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right).$$