Corrections to `ghm22@cam.ac.uk`*. Star* ($\star$) *indicates a harder question.*

This sheet builds on the last with more difficult contour integrals.

1 By considering the integral $\oint (z^2+1)^{-1}e^{ikz}\,\mathrm{d}z$ taken around a large semicircle, show that for real positive k

$$\int_{-\infty}^{\infty} \frac{\cos(kx)\,\mathrm{d}x}{x^2+1} = \pi e^{-k}$$

What is the value of the integral for $k \le 0$?

2 Let n be an integer greater than 3. Deduce the required sector angle α for a wedge-shaped contour in the complex plane that allows for the direct evaluation of

$$\int_0^{\infty} \frac{\mathrm{d}x}{1+x^n}.$$

Show that the integral evaluates to $\frac{\pi}{n\sin(\pi/n)}$.

3 Let a be a real non-zero constant. Show that

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{a^2+\sin^2\theta} = \frac{\pi}{a\sqrt{1+a^2}} \qquad (a>1).$$

4 By considering $f(z) = \exp\left(\frac{1}{2}iaz^2\right)$ and taking a wedge in the complex plane, show that

$$\int_0^{\infty} \cos\left(\frac{1}{2}ax^2\right)\mathrm{d}x = \sqrt{\frac{\pi}{4|a|}}.$$

5 Choose appropriate branch cuts to show that:

a. $$\int_0^{\infty} \frac{x^{-a}\,\mathrm{d}x}{x+1} = \frac{\pi}{\sin(\pi a)} \qquad (0<a<1),$$

b. $$\int_0^{\infty} \frac{(\log x)^2\,\mathrm{d}x}{1+x^2} = \frac{\pi^3}{8}.$$

6 Evaluate the Dirichlet integral,

$$\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x\,,$$

by considering $f(z) = \frac{e^{iz}}{z}$ integrated along a notched semicircular contour in the upper half plane with outer radius R and inner radius ϵ.

Note that this is an example of the *fractional residue theorem*: passing through an angle α counterclockwise around a simple pole yields a value of $i\alpha\,\mathrm{Res}(f, z_0)$.

[Hint: Evaluate the integral along the ϵ *notch by directly parameterising it by* $z = \epsilon e^{i\theta}$ *with appropriate limits.]*

7 By choosing an appropriate rectangular contour whose height is chosen to exploit the periodicity of the exponential function, evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,\mathrm{d}x \qquad (0<a<1).$$

Why does a standard semicircular contour fail for this integrand?

8 The reciprocal Gamma function can be written using Hankel's loop integral

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^t t^{-z} \, \mathrm{d}t$$

where C is an open contour starting at $-\infty - i\epsilon$, wrapping counterclockwise around the origin, and returning to $-\infty + i\epsilon$. For real $z < 1$, show how this open path completely evades the branch cut of t^{-z}, and use it to prove the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

9 By considering the integral of $(\cot z)/(z^2 + \pi^2 a^2)$ around a large square with vertices at $\pm(N + \frac{1}{2})\pi \pm i(N + \frac{1}{2})\pi$, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$$

provided that ia is not an integer. By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

Find an expression for $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ and take the limit as $a \to 0$ to deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.