NST IB: Mathematical Methods II — Sheet 8: Fourier transform methods

Corrections to `ghm22@cam.ac.uk`*. Star ($\star$) indicates a harder question.*

1 Find the function $f(x)$ that has Fourier transform $\tilde{f}(k) = (1+k^4)^{-1}$.

2 Find the function $f(x)$ that has Fourier transform $\tilde{f}(k) = (k^2+2ik+5)^{-1}$ by using contour integration to evaluate the inverse transform.

3 The displacement $x(t)$ of a critically damped harmonic oscillator satisfies the differential equation $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = \delta(t)$, where $\gamma > 0$. Given the causality condition $x(t) = 0$ for $t < 0$, show by Fourier transform methods that for $t > 0$,

$$x(t) = te^{-\gamma t}.$$

4 The motion of an overdamped harmonic oscillator subject to an impulsive force is described by the equation $\ddot{x} + 2\gamma\dot{x} + p^2 x = \delta(t)$ for a function $x(t)$ and constants $\gamma > p > 0$. Given that $x = 0$ for $t < 0$, show by Fourier transform methods that for $t > 0$,

$$x(t) = \frac{e^{-\gamma t}}{\sqrt{\gamma^2 - p^2}} \sinh\left(\sqrt{\gamma^2 - p^2}\, t\right).$$

5 A function $u(x,t)$, defined for all $t \geq 0$, satisfies the diffusion equation $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}$ subject to the initial conditions

$$u(x,0) = \begin{cases} -e^{x} & x < 0 \\ 0 & x = 0 \\ e^{-x} & x > 0 \end{cases}$$

Using Fourier transform methods, show that for $t > 0$

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{k e^{-\lambda k^2 t} \sin(kx)}{1+k^2}\, dk.$$

Suppose now that $u(x,t)$ is defined only for $x \geq 0$, that $u(0,t) = 0$ for all $t \geq 0$, and that the initial condition is $u(x,0) = e^{-x}$ for $x > 0$. Write down the solution of this modified problem.

6 A function $u(x,t)$ satisfies the diffusion equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $-\infty < x < \infty$ and $t > 0$, subject to the initial condition

$$u(x,0) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ 0 & x < 0 \end{cases}$$

By applying a Fourier transform with respect to x, show that for $t > 0$, the solution can be expressed as

$$u(x,t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin(kx)}{k} e^{-k^2 t}\, \mathrm{d}k\,.$$

7 The displacement $u(x,t)$ of an infinite string satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $-\infty < x < \infty$ and $t > 0$. The string is initially at rest, $u(x,0) = 0$, but is given an initial velocity profile $\frac{\partial u}{\partial t} = g(x)$. Use Fourier transform methods to derive the solution

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\, \mathrm{d}\xi\,.$$

8 Use Fourier transforms to solve the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-\xi|} y(\xi)\,\mathrm{d}\xi = \frac{1}{1+x^2}$$

for the unknown function $y(x)$. Express your answer as a real integral in terms of the transform variable k.

[Hint: You worked out the Fourier transform of $e^{-|x|}$ in a Michaelmas question sheet.]

9 An infinite structural railroad track resting on an elastic embankment can be modeled as an Euler-Bernoulli beam on an elastic foundation. Under a static point load at the origin, the vertical deflection $u(x)$ satisfies the fourth-order equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + 4\alpha^4 u = \delta(x),$$

where $\alpha > 0$ depends on the stiffness of the beam and the foundation. Show by Fourier transform methods that the deflection profile is given by

$$u(x) = \frac{1}{4\alpha^3} e^{-\alpha|x|} \left(\cos(\alpha x) + \sin(\alpha|x|)\right),$$

using the residue theorem to evaluate the inverse transform over the poles in the complex k-plane.

10 The Fourier transform of $y(t)$ is given by

$$\tilde{y}(\omega) = \frac{-\omega \tilde{f}(\omega)}{\omega^3 - i\omega^2 + 4\omega - 4i},$$

where $\tilde{f}(\omega)$ is the Fourier transform of the function $f(t)$, and both $y(t)$ and $f(t)$ vanish as $t \to \pm\infty$.

Determine the third order differential equation that governs $y(t)$. Find $f(t)$, valid for all t, for the case

$$\tilde{f}(\omega) = \frac{-i}{\omega - i}.$$

Use an inverse Fourier transform to determine $y(t)$ for this $\tilde{f}(\omega)$, valid for all t. Sketch the behaviour of $y(t)$.

11 Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (a+b)\frac{\mathrm{d}y}{\mathrm{d}t} + aby = f(t),$$

with the constants a and b such that $a > b > 0$. The solution $y(t)$ and its derivatives may be assumed to tend to zero as $t \to \pm\infty$.

Derive an equation relating $\tilde{y}$ to $\tilde{f}$, carefully justifying all steps in your calculation. Deduce that

$$y(t) = \int_{-\infty}^{\infty} f(s) G(t-s)\,\mathrm{d}s,$$

and find the function $G(t)$ by inverting $\tilde{G}(k)$.

In the case $f(t) = e^{-|t|}$ evaluate $\tilde{f}(k)$. Assuming that $a \neq 1$ and $b \neq 1$, deduce an expression for $\tilde{y}(k)$ and invert to deduce $y(t)$.