NST IB: Mathematical Methods I Sheet 2: Suffix notation & vector calculus

Corrections to `ghm22@cam.ac.uk`. *Star (⋆) indicates a harder question.*

This sheet should be largely revision from 1A, especially if you did the B course. Fluency with suffix notation and the summation convention is *incredibly* important for future topics in both this course and your later studies. If you're struggling after these questions, we can find more to practise.

1 In the following, the indices i, j, k, l, m take the values $1, 2, 3$, and the summation convention applies. In particular, let n be a unit vector.

a. Simplify the following expressions:

$$\delta_{ij}a_j, \quad \delta_{ij}\delta_{jk}, \quad \delta_{ij}\delta_{ji}, \quad \delta_{ij}n_in_j,$$

$$\epsilon_{ijk}\delta_{jk}, \quad \epsilon_{ijk}\epsilon_{ijl}, \quad \epsilon_{ijk}\epsilon_{ikj}, \quad \epsilon_{ijk}(a \times b)_k.$$

b. For each of the following equations, either give the equivalent in vector or matrix notation or explain why the equation is invalid:

$$x_i = a_ib_kc_k + d_i, \quad x_i = a_jb_i + c_kd_ie_kf_j, \quad u = \epsilon_{jkl}v_kw_lx_j,$$

$$\epsilon_{ijk}x_jy_k\epsilon_{ilm}x_ly_m = \mu, \quad A_{ik}B_{kl} = T_{ik}\delta_{kl}, \quad x_k = A_{ki}B_{ji}y_j.$$

c. Write the following equations in suffix notation using the summation convention:

$$(x + \mu y) \cdot (x - \mu y) = \kappa,$$

$$\mathbf{x} = |\mathbf{a}|^2\mathbf{b} - |\mathbf{b}|^2\mathbf{a}, \quad (2\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{a} + \mathbf{b}) = \lambda.$$

d. Given that $A_{ij} = \epsilon_{ijk}a_k$ (for all i, j), show that $2a_k = \epsilon_{kij}A_{ij}$ (for all k).

e. Show that $\epsilon_{ijk}s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).

f. Given that $N_{ij} = \delta_{ij} - \epsilon_{ijk}n_k + n_in_j$ and $M_{ij} = \delta_{ij} + \epsilon_{ijk}n_k$, show that $N_{ij}M_{jk} = 2\delta_{ik}$.

2 Use suffix notation and the identity for $\epsilon_{ijk}\epsilon_{ilm}$ (which you should commit to memory) to prove the vector triple product identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Use suffix notation to expand the curl of a cross product, $\nabla \times (\mathbf{F} \times \mathbf{G})$. Express your final answer in standard vector notation.

3 Let $\mathbf{r}$ be the position vector with components x_i, and let $r = |\mathbf{r}|$. Let $\mathbf{a}$ be a constant vector, T_{ij} be a constant symmetric tensor, and $f(r)$ be a differentiable scalar function.

a. Using suffix notation, evaluate:

$$\frac{\partial r}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 r}{\partial x_i \partial x_j}.$$

b. Show that:

$$\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \quad \text{and} \quad \nabla \cdot (f(r)\mathbf{r}) = 3f(r) + rf'(r).$$

c. Define the scalar field $\Phi = T_{ij}x_ix_j$. Find an expression for the vector field $\mathbf{G} = \nabla\Phi$ in suffix notation, and show that $\nabla \cdot \mathbf{G} = 2T_{ii}$.

d. Evaluate the components of the vector field $\mathbf{H} = \nabla \times (f(r)\mathbf{a} \times \mathbf{r})$ using suffix notation, and simplify your result into standard vector notation.

4 A fluid flow has the constant velocity vector (in Cartesian coordinates)

$$\mathbf{v}(\mathbf{r}) = (0, 0, W).$$

Explicitly calculate the volume flux of fluid,

$$Q = \int \mathbf{v} \cdot \mathrm{d}\mathbf{S}$$

flowing across (a) the open hemispherical surface $r = a$, $z \geq 0$, and (b) the disc $r \leq a$, $z = 0$. Verify that the divergence theorem holds.

5 For a surface S enclosing a volume V, apply the divergence theorem to a vector field $\mathbf{F} = \mathbf{a}p$, where $\mathbf{a}$ is an arbitrary constant vector and $p(\mathbf{r})$ is a scalar field. Deduce that

$$\int_V \nabla p \, \mathrm{d}V = \oint_S p \, \mathrm{d}\mathbf{S} \, .$$

6 Consider the vector field $\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}$, where $\mathbf{r}$ is the position vector and $r = |\mathbf{r}|$. Show that $\nabla \cdot \mathbf{F} = 0$ everywhere except at the origin. Evaluate the surface integral $\oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over the surface of a sphere of radius R centred at the origin. Is this result consistent with the divergence theorem?

7 A time-independent magnetic field $\mathbf{B}(\mathbf{r})$ is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{\mathbf{e}_z \times \mathbf{r}}{x^2 + y^2}$$

where μ_0 is the magnetic permeability and I is a constant. Using Cartesian coordinates, calculate the electric current density $\mathbf{J}$ given by the steady Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Also evaluate $\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r}$, where C is a circle of radius a in the plane $z = 0$ and centred on $x = y = 0$. Discuss whether Stokes's theorem applies in this situation.

8 Show that, in Cartesian coordinates,

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

This vector identity remains true for all coordinate systems; however, for non-Cartesian coordinates,

$$(\nabla^2 \mathbf{F})_i \neq \nabla^2 F_i.$$

Why is this the case? Illustrate this point by evaluating $\nabla^2 \mathbf{F}$ for $\mathbf{F} = f(\rho)\mathbf{e}_\phi$ in cylindrical polar coordinates (ρ, ϕ, z) and comparing the result with $\nabla^2 f$.

9 Find the general circularly symmetric solution to the fourth-order equation

$$\nabla^4 \Psi \equiv \nabla^2(\nabla^2 \Psi) = 0.$$

Hint: use plane polar coordinates (ρ, ϕ), and do not be too eager to expand everything out.

Find those circularly symmetric solutions in the unit disc that are equal to unity at the centre $\rho = 0$ and vanish on the boundary $\rho = 1$. Give a further condition to render the solution unique.

10 Let ϕ and ψ be scalar fields defined in a volume V bounded by a closed surface S. By applying the divergence theorem to the vector field $\mathbf{A} = \phi\nabla\psi$, prove Green's first identity,

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,\mathrm{d}V = \oint_S \phi\nabla\psi\cdot\mathrm{d}\mathbf{S}\,.$$

Hence, derive Green's second identity,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,\mathrm{d}V = \oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathrm{d}\mathbf{S}\,.$$

We will use these later when we come to Green's function solutions to PDEs.