NST IB: Mathematical Methods I   Sheet 3: Curvilinear coordinate systems

Corrections to `ghm22@cam.ac.uk`*. Star (⋆) indicates a harder question.*

Curvilinear coordinates are a new topic, but this sheet also has a few questions revising solutions of partial differential equations via separation of variables. This won't be new if you did course B last year. If you did course B, you may wish to skip those PDE questions in favour of the hard questions at the end.

1 Let q_i be orthogonal curvilinear coordinates. By requiring that $\nabla\phi \cdot \mathrm{d}\mathbf{x} = \partial\phi/\partial x$ and applying the chain rule, show that,

$$\nabla\phi = \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial q_i}\mathbf{e}_i,$$

carefully defining the quantities h_i and $\mathbf{e}_i$.

2 Parabolic coordinates (u, v, ϕ) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \frac{1}{2}(u^2 - v^2)$$

Show that the surfaces of constant u, and those of constant v, are surfaces obtained by rotating parabolae about the z-axis. What are the surfaces of constant ϕ? Show that the coordinate surfaces intersect at right angles and hence that these coordinates are orthogonal. Find the scale factors (h_u, h_v, h_ϕ) defined by

$$|\mathrm{d}\mathbf{r}|^2 = h_u^2\,\mathrm{d}u^2 + h_v^2\,\mathrm{d}v^2 + h_\phi^2\,\mathrm{d}\phi^2\,.$$

Hence obtain the Laplacian in these coordinates using the formula

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right]$$

3 Elliptic cylindrical coordinates (u, v, z) are defined in terms of the Cartesian coordinates (x, y, z) by

$$x = a\cosh u\cos v,$$
$$y = a\sinh u\sin v,$$
$$z = z,$$

where $u \geq 0$, $0 \leq v < 2\pi$, $-\infty < z < \infty$. Sketch the surfaces of constant u and the surfaces of constant v. Show that the coordinate surfaces intersect at right angles and thus these coordinates are orthogonal coordinates.

Find the metric coefficients (h_u, h_v, h_z) defined by

$$\mathrm{d}r^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = h_u^2\,\mathrm{d}u^2 + h_v^2\,\mathrm{d}v^2 + h_z^2\,\mathrm{d}z^2\,,$$

and show that $h_u = h_v$.

By calculating the relevant volume element $\mathrm{d}V = h_uh_vh_z\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z$, evaluate the integral

$$I = \int_V (x^2 + y^2)\,\mathrm{d}V$$

over the region V enclosed by the elliptic cylinder $u = U$ and the planes $z = 0$ and $z = H$.

4 Consider the two-stage transformation of Cartesian coordinates (x, y, z) given by

$$x = ax', \quad y = by', \quad z = cz'$$

$$x' = r' \sin\theta' \cos\phi', \quad y' = r' \sin\theta' \sin\phi', \quad z' = r' \cos\theta'$$

where a, b and c are positive constants. Calculate the Jacobian matrices of the transformations $(x, y, z) \mapsto (x', y', z')$ and $(x', y', z') \mapsto (r', \theta', \phi')$ and verify explicitly that

$$\frac{\partial(x, y, z)}{\partial(r', \theta', \phi')} = \frac{\partial(x, y, z)}{\partial(x', y', z')} \frac{\partial(x', y', z')}{\partial(r', \theta', \phi')}$$

Are the coordinates (r', θ', ϕ') orthogonal? What range of these coordinates is required to cover the interior of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,?$$

Express the volume element in coordinates (r', θ', ϕ') and hence calculate the volume of the ellipsoid.

5 Oblate spheroidal coordinates (ξ, η, ϕ) are related to Cartesian coordinates by

$$x = a\cosh\xi \sin\eta \cos\phi, \quad y = a\cosh\xi \sin\eta \sin\phi, \quad z = a\sinh\xi \cos\eta$$

where $\xi \geq 0$, $0 \leq \eta \leq \pi$, and $0 \leq \phi < 2\pi$.

Identify the geometric surfaces of constant ξ and constant η. Determine the metric scale factors h_ξ, h_η, and h_ϕ, and write down the expression for the gradient $\nabla\Psi$ in this coordinate system.

6 In a Cartesian coordinate system (x_1, x_2, x_3), A is the point $(0, 0, -1)$, B is the point $(0, 0, 1)$ and P is an arbitrary point (x_1, x_2, x_3). In a curvilinear coordinate system, the coordinates of P are specified by

$$u_1 = \frac{1}{2}(r_1 + r_2), \quad u_2 = \frac{1}{2}(r_1 - r_2), \quad u_3 = \phi$$

where r_1 and r_2 are the distances AP and BP respectively and ϕ is the angle between the planes ABP and $x_2 = 0$. Show that $x_3 = u_1 u_2$ and that the distance ρ from P to the x_3-axis is equal to $\sqrt{(u_1^2 - 1)(1 - u_2^2)}$.

Next evaluate $\partial x_i/\partial u_j$ (with $i, j = 1, 2, 3$). Deduce that the curvilinear coordinates are orthogonal and sketch the coordinate surfaces. Show that the metric coefficients are

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}}, \quad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)}.$$

Show that if a function satisfies Laplace's equation and is independent of u_2 and u_3 then it has the form

$$\Psi = a + b\log\left(\frac{u_1 - 1}{u_1 + 1}\right)$$

for constants a and b.

7 Show that the solution of Laplace's equation, $\nabla^2\Phi = 0$, in the region $0 < x < a$, $0 < y < b$, $0 < z < c$ with $\Phi = 1$ on the surface $z = 0$ and $\Phi = 0$ on the other surfaces, is

$$\Phi(x, y, z) = \frac{16}{\pi^2} \sum_{\text{odd } m} \sum_{\text{odd } n} \frac{\sin(m\pi x/a)\sin(n\pi y/b)\sinh(k(c - z))}{mn\sinh(kc)}$$

where $k^2 = m^2\pi^2/a^2 + n^2\pi^2/b^2$.

8 The temperature distribution $\theta(x,t)$ along a thin bar of length L satisfies the one-dimensional diffusion equation

$$\frac{\partial\theta}{\partial t} = \nu\frac{\partial^2\theta}{\partial x^2}$$

where the diffusivity ν is a constant, t denotes time and x is the distance from one of the ends. Find $\theta(x,t)$ if the bar is insulated at each end (i.e. if $\partial\theta/\partial x = 0$ at each end), and if the initial temperature distribution is given by

$$\theta(x,0) = 2\theta_0\cos^2\left(\frac{\pi x}{L}\right),$$

where θ_0 is a constant. For large times what is the temperature distribution in the bar? Comment.

9⋆ Derive the expression for the divergence of a vector field $\mathbf{A} = \sum_i A_i\mathbf{e}_i$ in general orthogonal curvilinear coordinates:

$$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3) + \frac{\partial}{\partial q_2}(A_2h_3h_1) + \frac{\partial}{\partial q_3}(A_3h_1h_2)\right]$$

by considering the net flux of $\mathbf{A}$ out of an infinitesimal volume element $\mathrm{d}V = h_1h_2h_3\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3$.

[Hint: If you're struggling with this, try it in Cartesian coordinates first.]

10⋆ Show that the $\mathbf{e}_1$-component of the curl of a vector field $\mathbf{A}$ in orthogonal curvilinear coordinates is given by

$$(\nabla\times\mathbf{A})_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial q_2}(h_3A_3) - \frac{\partial}{\partial q_3}(h_2A_2)\right]$$

by applying Stokes' Theorem ($\oint_C \mathbf{A}\cdot\mathrm{d}\mathbf{x} = \iint_S(\nabla\times\mathbf{A})\cdot\mathrm{d}\mathbf{S}$) to an infinitesimal rectangular loop lying within a surface of constant q_1.

[Hint: As before, if struggling, try Cartesian first.]