NST IB: Mathematical Methods I — Sheet 4: Green's functions for ODEs

Corrections to `ghm22@cam.ac.uk`*. Star (⋆) indicates a harder question.*

1 Define $\delta_\epsilon(x)$ for $\epsilon > 0$ by

$$\delta_\epsilon(x) = \begin{cases} (x+\epsilon)/\epsilon^2, & -\epsilon < x < 0, \\ (\epsilon - x)/\epsilon^2, & 0 \le x < \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

a. Evaluate $\int_{-\infty}^{\infty} \delta_\epsilon(x)\,\mathrm{d}x$.

b. Argue that for a "nice" function f and a constant ξ

$$\lim_{\epsilon \to 0+} \int_{-\infty}^{\infty} \delta_\epsilon(x-\xi) f(x)\,\mathrm{d}x = f(\xi).$$

Hint: Consider the substitution $x - \xi = \epsilon t$.

c. Sketch $\delta_\epsilon(x)$ and comment.

2 Starting from the definition that $\delta(x)$ is the generalized function such that for all "nice" functions $f(x)$

$$\int_{-\infty}^{\infty} \delta(x-\xi) f(x)\,\mathrm{d}x = f(\xi),$$

show that, for constant $a \neq 0$, $\delta(ax) = \frac{1}{|a|}\delta(x)$.

3 Evaluate

$$\int_{-\infty}^{\infty} |x|\delta(x^2 - a^2)\,\mathrm{d}x$$

where a is a non-zero constant by substituting $t = x^2$ and taking care over the limits of integration.

Given a function g with zeroes at x_i, derive a general expression for

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\,\mathrm{d}x\,.$$

Does your answer agree with the previous part of this question?

4 The derivative of the Dirac delta function, $\delta'(x)$, is defined by performing integration by parts on the expression,

$$\int_{-\infty}^{\infty} \delta'(x-\xi) f(x)\,\mathrm{d}x\,,$$

where f is a sufficiently 'nice' function. Evaluate this integral to obtain a result in terms of the derivative of f.

a. Show that because $\delta(x)$ is an even 'function' ($\delta(-x) = \delta(x)$), its derivative is odd; that is $\delta'(-x) = -\delta'(x)$.

b. Evaluate the integral

$$\int_{-\infty}^{\infty} x^4 \delta''(x-2)\,\mathrm{d}x\,.$$

c. By treating the Heaviside step function $H(x)$ as the distribution satisfying $H'(x) = \delta(x)$, use integration by parts to evaluate

$$\int_{-\infty}^{\infty} H(x)\delta'(x)\,\mathrm{d}x\,.$$

Is your result well-defined?

5 Show that the equation

$$y'' + py' + qy = f(x),$$

where p and q are constants, can be written in the form

$$y' - by = z, \quad z' - az = f,$$

for suitable choices of the constants a and b. Solve these first-order equations using integrating factors, subject to the initial conditions $y(0) = y'(0) = 0$, to obtain the solution

$$y(x) = e^{bx} \int_0^x \int_0^\eta f(\xi) e^{-a\xi} e^{(a-b)\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

By changing the order of integration and carrying out the integration with respect to η, show that

$$y(x) = \frac{1}{a-b} \int_0^x f(\xi) \left[e^{a(x-\xi)} - e^{b(x-\xi)} \right] \mathrm{d}\xi$$

and interpret this result.

6 The differential equation

$$y'' + y = H(x) - H(x - \epsilon),$$

where H is the Heaviside step function and ϵ is a positive parameter, represents a simple harmonic oscillator subject to a constant force for a finite time. By solving the equation in the three intervals of x separately and applying appropriate matching conditions, show that the solution that vanishes for $x < 0$ is

$$y = \begin{cases} 0, & x < 0, \\ 1 - \cos x, & 0 < x < \epsilon, \\ \cos(x - \epsilon) - \cos x, & x > \epsilon. \end{cases}$$

Hence show that the solution of

$$y'' + y = \frac{H(x) - H(x - \epsilon)}{\epsilon}$$

that vanishes for $x < 0$ agrees, in the limit $\epsilon \to 0$, with the appropriate solution of $y'' + y = \delta(x)$, namely $y = H(x)\sin x$.

7 The function $G(x, \xi)$ is defined by

$$G(x, \xi) = \begin{cases} x(\xi - 1), & 0 \le x \le \xi, \\ \xi(x - 1), & \xi \le x \le 1. \end{cases}$$

If $f(x)$ is continuous for $0 \le x \le 1$ and

$$y(x) = \int_0^1 f(\xi) G(x, \xi)\,\mathrm{d}\xi\,,$$

show by direct calculation that $y''(x) = f(x)$ and find $y(0)$ and $y(1)$. *Hint: use the definition of $G(x, \xi)$ to write $y(x)$ as the sum of two integrals, one with $\xi \le x$ and the other with $x \le \xi$.*

8 Construct the Green's function for the boundary value problem $y'' - y = f(x)$ subject to $y(0) = y(1) = 0$. Use it to solve $y'' - y = x^2$.

Solve the boundary value problem $y'' + \omega^2 y = x$ with $y'(0) = y(\pi/\omega) = 0$ using a Green's function method.

9 Green's functions can also be constructed for *initial value problems*. What is the key difference that arises when applying the boundary values?

Construct Green's functions for the following two initial value problems, and use them to solve with the particular integral:

a. $y'' + \alpha y' = e^{-\beta x}$ with $x \geq 0$ and $y(0) = y'(0) = 0$,

b. $y'''' = f(x)$ with $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

10⋆ Consider the boundary value problem

$$y'' = f(x), \quad 0 < x < 1,$$

subject to the Neumann boundary conditions $y'(0) = 0$ and $y'(1) = 0$.

By integrating the differential equation from 0 to 1, show that a solution can only exist if $f(x)$ satisfies the compatibility condition $\int_0^1 f(x)\,\mathrm{d}x = 0$. Explain why a standard Green's function satisfying $G'' = \delta(x - \xi)$ cannot exist for these boundary conditions.

To avoid this, we define a *modified Green's function* $G_m(x, \xi)$ satisfying

$$\frac{\mathrm{d}^2 G_m}{\mathrm{d}x^2} = \delta(x - \xi) - 1.$$

Find $G_m(x, \xi)$ subject to $y'(0) = y'(1) = 0$. To ensure uniqueness, impose the further normalization condition $\int_0^1 G_m(x, \xi)\,\mathrm{d}x = 0$.