

Corrections to *ghm22@cam.ac.uk*. Star (\star) indicates a harder question.

- 1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

- 2 For a Hermitian matrix H , explain how to construct a unitary matrix U such that $U^\dagger H U = D$, where D is a real diagonal matrix. Illustrate the procedure with the matrix

$$H = \begin{bmatrix} 4 & 3i \\ -3i & -4 \end{bmatrix}.$$

- 3 An anti-Hermitian matrix A is one for which $A^\dagger = -A$. What can be said about the eigenvalues of A ?

If S is real symmetric and T is real antisymmetric, show that $T \pm iS$ are anti-Hermitian. Deduce that $\det(T + iS - I) \neq 0$. Show that the matrix

$$U = (I + T + iS)(I - T - iS)^{-1}$$

is unitary.

- 4 Let λ_i be the eigenvalues of an $n \times n$ Hermitian matrix A . Show that,

$$\text{Tr } A^k = \sum_{i=1}^n \lambda_i^k \quad \text{and} \quad \det A^k = \prod_{i=1}^n \lambda_i^k,$$

and also that,

$$\det(\exp A) = \exp(\text{Tr } A),$$

where $\exp A$ is defined in the standard way,

$$\det A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Hence conclude that the exponential of an arbitrary real antisymmetric matrix acts as a rotation in \mathbb{R}^n .

- 5 Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue then so is λ^* . Hence argue that the eigenvalues of a 3×3 real orthogonal matrix R must be a selection from $+1, -1$ and $e^{\pm i\alpha}$. Verify that $\det R = \pm 1$. What is the effect of R on vectors orthogonal to an eigenvector with eigenvalue ± 1 ?

- 6 Let H be an $n \times n$ Hermitian matrix with n distinct eigenvalues $\{\lambda_i\}$ and orthonormal eigenvectors $\{e_i\}$. Now consider the slightly perturbed matrix $H + \delta H$, where δH is small and

Hermitian. Let the eigenvalues and orthonormal eigenvectors of $H + \delta H$ be $\{\lambda_i + \delta\lambda_i\}$ and $\{e_i + \delta e_i\}$ respectively. By working to first order in the small quantities, show that

$$\delta\lambda_i = e_i^\dagger(\delta H)e_i, \quad \delta e_i = \sum_{j \neq i} \frac{e_j^\dagger(\delta H)e_i}{\lambda_i - \lambda_j} e_j.$$

Why is it permissible to omit any contribution to δe_i parallel to e_i ?

[Hint: Write out the eigenvector equation and the orthonormality condition for the eigenvectors of the perturbed matrix. Expand, neglect products of small quantities, and simplify. Use the fact that $\{e_i\}$ is a basis.]

- 7 Find the eigenvalues and normalized eigenvectors of the symmetric matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}.$$

Describe the related quadratic surfaces.

- 8 A complex square matrix A is called *normal* if it commutes with its Hermitian conjugate, so that $AA^\dagger = A^\dagger A$.

Prove that A is normal if and only if $|A\mathbf{x}| = |A^\dagger\mathbf{x}|$ for all vectors \mathbf{x} .

Show that if \mathbf{x} is an eigenvector of a normal matrix A with eigenvalue λ , then \mathbf{x} is also an eigenvector of A^\dagger with eigenvalue λ^* . Hence, prove that eigenvectors corresponding to distinct eigenvalues of a normal matrix must be mutually orthogonal.

- 9 Let H be an $n \times n$ Hermitian matrix with eigenvalues ordered such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. The Rayleigh quotient for a non-zero vector $\mathbf{x} \in \mathbb{C}^n$ is defined as:

$$R(\mathbf{x}) = \frac{\mathbf{x}^\dagger H \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}}$$

By expanding \mathbf{x} in terms of an orthonormal basis of eigenvectors of H , prove the Rayleigh–Ritz variational principle:

$$\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n$$

Under what conditions regarding \mathbf{x} do the upper and lower bounds hold with equality?

- 10 Let H be an $n \times n$ Hermitian matrix. Show that the matrix $(H + iI)$ is always invertible.

The *Cayley transform* of H is defined as the matrix:

$$U = (H - iI)(H + iI)^{-1}$$

Prove that U is a unitary matrix. Assuming that $(I - U)$ is invertible, find an explicit expression for H in terms of U .

- 11 Let A be an invertible $n \times n$ complex matrix.

Show that $A^\dagger A$ is a Hermitian, positive-definite matrix. Let P be the unique positive-definite square root of $A^\dagger A$, such that $P^2 = A^\dagger A$.

Show that the matrix $U = AP^{-1}$ is unitary. Deduce the Polar Decomposition: any invertible complex matrix can be uniquely written as the product of a unitary matrix and a positive-definite Hermitian matrix.